Synthesizing Cubes to Satisfy a Given Intersection Pattern*

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ABSTRACT

In two-level logic synthesis, the typical input specification is a set of minterms defining the on set and a set of minterms defining the don't care set of a Boolean function. The problem is to synthesize an optimal set of product terms, or cubes, that covers all the minterms in the on set and some of the minterms in the don't care set. In this paper, we consider a different specification: instead of the on set and the don't care set, we are given a set of numbers, each of which specifies the number of minterms covered by the intersection of one of the subsets of a set of λ cubes. We refer to the given set of numbers as an *intersection pattern*. The problem is to deterimine whether there exists a set of λ cubes to satisfy the given intersection pattern and, if it exists, to synthesize the set of cubes. We show a necessary and sufficient condition for the existence of λ cubes to satisfy a given intersection pattern. We also show that the synthesis problem can be reduced to the problem of finding a non-negative solution to a set of linear equalities and inequalities.

1. INTRODUCTION

Two-level logic synthesis is a well-developed and mature topic [1, 2]. The typical input specification for a two-level synthesis problem is the *on* set and the *don't care* set (or in some cases, the *off* set) of a Boolean function. The on set and the don't care set consist of minterms that define when the function evaluates to one and when its evaluation can be either zero or one, respectively. The problem is to synthesize an optimal set of product terms, or *cubes*, that covers all the minterms in the on set and some of the minterms in the don't care set.

In this work, we consider a related yet different problem pertaining to the synthesis of a set of cubes. A set of cubes, besides defining a Boolean function, also defines a set of numbers, each of which corresponds to the number of minterms covered by the intersection of one of the subsets of the set of cubes. For example, given a set of three cubes on four variables x_0, x_1, x_2, x_3, which are $c_0 = x_0x_1$, $c_1 = x_2$, and $c_2 = x_1x_3$, the numbers of minterms covered by c_0, c_1, c_2, c_0c_1, c_0c_2, c_1c_2, and $c_0c_1c_2$ are 4, 8, 4, 2, 2, 2, and 1, respectively. We refer to this set of numbers as an *intersection pattern*.

Given a set of cubes, it is trivial to get its intersection pattern. However, it is nontrivial to answer the reverse problem: given a set of numbers that corresponds to an intersection pattern of λ cubes, how can one synthesize a set of λ cubes to satisfy the given intersection pattern, or prove that there is no solution to the given intersection pattern? We will call this the *λ-cube intersection problem.* It is what we intend to solve in this paper.

Definition 1
Define $V(f)$ to be the number of minterms contained in a Boolean function f. □

Example 1
In a 3-cube intersection problem on 4 variables x_0, x_1, x_2, x_3, if we are given the intersection pattern as

$$V(c_0) = 4, V(c_1) = 8, V(c_2) = 4,$$
$$V(c_0c_1) = V(c_0c_2) = V(c_1c_2) = 2, V(c_0c_1c_2) = 1,$$

we can synthesize cubes $c_0 = x_0x_1$, $c_1 = x_2$, and $c_2 = x_1x_3$ to satisfy the intersection pattern. □

We are interested in the λ-cube intersection problem since it pertains to synthesis for probabilistic computation, a new paradigm that we have advocated [3]. In this paradigm, digital circuits are designed to transform a set of input probabilities, encoded by random bit streams, into output probabilities, also encoded by random bit streams [3]. A fundamental problem in this context is how to synthesize combinational logic that takes independent inputs with probability 0.5 of being one and generates other probabilities as outputs. For example, we can use the combinational circuit shown in Figure 1 to generate an output probability $\frac{3}{8}$ from three independent input probabilities 0.5.

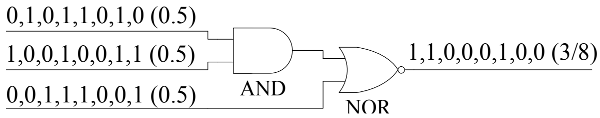

Figure 1: An AND gate followed by a NOR gate transforms three independent random inputs of probability 0.5 of being one into an random output of probability $\frac{3}{8}$ of being one. The inputs and output of the circuit are random bit streams. The numbers in the parentheses denote the probabilities.

For combinational logic with n inputs with each input independently having probability 0.5 of being one, each input combination has probability of $\frac{1}{2^n}$ of occurring. If the Boolean function contains exactly m minterms, then the probability that the output is one is $\frac{m}{2^n}$. Conversely, if we want to synthesize a probability $\frac{m}{2^n}$ $(0 \le m \le 2^n)$, we can simply implement it with a Boolean function of m minterms. However, there are $\binom{2^n}{m}$ Boolean functions that contain exactly m minterms and different functions have different implementation cost. This motivates a new problem in logic synthesis: if we want to synthesize a logic circuit such that it covers exactly m minterms, while which m minterms are covered does not matter, then how can we design an optimal logic circuit?

We focus on two-level implementation of logic circuit [1]. Minimizing the area of the two-level implementation is equivalent to minimizing the number of cubes of the sum-of-product (SOP) representation of a Boolean function [1]. Thus, the problem, which we will refer to as the *arithmetic two-level minimization problem*, can be formulated as:

Given the number of variables n for a Boolean function and an integer $0 \le m \le 2^n$, find a SOP Boolean expression with the minimum number of cubes that contains exactly m minterms.

For the arithmetic two-level minimization problem, our proposed solution is based on the inclusion-exclusion principle:

Given λ cubes $c_0, \dots, c_{\lambda-1}$, the number of minterms cover by the λ cubes is

$$V\left(\bigvee_{i=0}^{\lambda-1} c_i\right) = \sum_{i=0}^{\lambda-1} V(c_i) - \sum_{\substack{i,j:\\0\le i<j\le\lambda-1}} V(c_ic_j) + \sum_{\substack{i,j,k:\\0\le i<j<k\le\lambda-1}} V(c_ic_jc_k) - \cdots + (-1)^{\lambda-1} V\left(\prod_{i=0}^{\lambda-1} c_i\right). \tag{1}$$

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The inclusion-exclusion principle connects the arithmetic two-level minimization problem with the λ-cube intersection problem. Indeed, we intend to apply a search-based approach to solve the minimization problem. Initially, we will set λ to be a lower bound on the number of cubes to cover m minterms. Then we will test whether we can find λ cubes so that they cover m minterms. In order to do so, we will first construct an intersection pattern such that the sum of the elements in that pattern according to Equation (1) equals the target value m. Then, we need to check whether we can find λ cubes to satisfy that intersection pattern. If we find a solution to that instance of the λ-cube intersection problem, then we obtain an optimal solution to the arithmetic two-level minimization problem. If not, we will try another intersection pattern on λ cubes. After a number of unsuccessful trials, we will increase λ by one.

Example 2
Synthesize an optimal SOP Boolean expression on 4 *variables to cover* 11 *minterms.*

Since we cannot cover 11 *minterms with just* 1 *cube, the lower bound on the solution is* 2 *cubes. Thus, initially, we set* $\lambda = 2$. *For* $\lambda = 2$, *we first construct intersection pattern* $V(c_0)$, $V(c_1)$ *and* $V(c_0c_1)$, *so that* $V(c_0)+V(c_1)-V(c_0c_1) = 11$. *One intersection pattern is* $V(c_0) = 8$, $V(c_1) = 4$ *and* $V(c_0c_1) = 1$. *However, that* 2*-cube intersection problem has no solution. Thus, we will try other intersection patterns on* 2 *cubes which cover* 11 *minterms. Indeed, there are no intersection patterns on* 2 *cubes to cover* 11 *minterms. Then, we raise* λ *to* 3.

For $\lambda = 3$, *we first construct intersection pattern* $V(c_0)$, $V(c_1)$, $V(c_2)$, $V(c_0c_1)$, $V(c_0c_2)$, $V(c_1c_2)$ *and* $V(c_0c_1c_2)$, *so that*

$$\begin{aligned} V(c_0) + V(c_1) + V(c_2) - V(c_0c_1) - V(c_0c_2) \\ - V(c_1c_2) + V(c_0c_1c_2) = 11. \end{aligned}$$

One intersection pattern is $V(c_0) = 8$, $V(c_1) = 2$, $V(c_2) = 1$ *and* $V(c_0c_1) = V(c_0c_2) = V(c_1c_2) = V(c_0c_1c_2) = 0$. *For that* 3*-cube intersection problem, we could synthesize cubes* $c_0 = x_0$, $c_1 = \bar{x}_0x_1x_2$ *and* $c_2 = \bar{x}_0\bar{x}_1\bar{x}_2x_3$ *to satisfy the given intersection pattern. Thus, we get an optimal solution of* 3 *cubes to the original arithmetic two-level minimization problem.* $\square$

2. PRELIMINARIES

In this section, we will first introduce some basic definitions and then give a formal definition of the λ-cube intersection problem. Some of the basic definitions are adopted from [4].

The set of n *variables* of a Boolean function is denoted as $x_0, \ldots, x_{n-1}$. For a variable x, x and $\bar{x}$ are referred to as *literals*. A *Boolean product*, or *cube*, denoted by c, is a conjunction of literals such that x and $\bar{x}$ do not appear simultaneously. A *minterm* is a cube in which each of the n variables appear once, in either its complemented or uncomplemented form. If cube c_2 takes the value one whenever cube c_1 equals one, we say that cube c_1 *implies* cube c_2 and write as $c_1 \subseteq c_2$. If cube c_1 implies cube c_2, then we have $V(c_1) \leq V(c_2)$. If $c_1 \cdot c_2 = 0$, we say that cube c_1 and c_2 are *disjoint*.

If a cube c contains k literals ($0 \leq k \leq n$), then the number of minterms contained in the cube is $V(c) = 2^{n-k}$. Note that when a cube contains 0 literals, it is a special cube $c = 1$, which contains all minterms in the entire Boolean space. There is another special cube called *empty cube*, which is $c = 0$. The number of minterms contained in an empty cube is $V(c) = 0$. Thus, the number of minterms contained in a cube is in the set $S = \{s | s = 0 \text{ or } s = 2^k, k = 0, 1, \ldots, n\}$.

To make the representation compact, we use the following definitions.

Definition 2
Given two integers A *and* B, *let their binary representation be* $A = \sum_{i=0}^{k-1} a_i 2^i$ *and* $B = \sum_{i=0}^{k-1} b_i 2^i$, *where* $a_i, b_i \in \{0, 1\}$. *We write* $A \succeq B$ *when for any* $0 \leq i \leq k-1$, $a_i \geq b_i$. $\square$

Definition 3
For a cube c, *define* $c^0 = 1$ *and* $c^1 = c$. *Given a set of* λ *cubes* $c_0, \ldots, c_{\lambda-1}$ *and an integer* $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$, *where* $\gamma_i \in \{0, 1\}$, *define* C^Γ *to be the intersection of a subset of cubes* c_i *with* $\gamma_i = 1$, *i.e.,* $C^\Gamma = \prod_{i=0}^{\lambda-1} c_i^{\gamma_i}$. $\square$

Definition 4
Given an integer $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$, *where* $\gamma_i \in \{0, 1\}$, *define* $B(\Gamma)$ *to be the number of ones in the binary representation of* Γ, *i.e.,* $B(\Gamma) = \sum_{i=0}^{\lambda-1} \gamma_i$. $\square$

With the above definition, we can more formally define the λ-cube intersection problem as follows:

Given $n > 0$, $\lambda > 0$, *and* $2^\lambda - 1$ *numbers* $v_1, v_2, \ldots v_{2^\lambda - 1} \in S = \{s | s = 0 \text{ or } s = 2^k, k = 0, 1, \ldots, n\}$, *determine whether there exists a set of* λ *cubes* $c_0, \ldots, c_{\lambda-1}$ *on* n *variables* $x_0, \ldots, x_{n-1}$, *such that for any* $1 \leq \Gamma \leq 2^\lambda - 1$, $V(C^\Gamma) = v_\Gamma$.

We refer to the vector of numbers $(v_1, \ldots, v_{2^\lambda - 1})$ as *an intersection pattern* on λ cubes, or simply as an intersection pattern. If a set of λ cubes $c_0, \ldots, c_{\lambda-1}$ satisfies the property that for any $1 \leq \Gamma \leq 2^\lambda - 1$, $V(C^\Gamma) = v_\Gamma$, then we say that the set of cubes satisfies the intersection pattern $(v_1, \ldots, v_{2^\lambda - 1})$.

Since it is more meaningful to consider a set of nonempty cubes $c_0, \ldots, c_{\lambda-1}$, we assume that for any $0 \leq i \leq \lambda - 1$, $v_{2^i} = V(c_i) > 0$. Further, notice that $V(C^0) = 2^n$. We let $v_0 = 2^n$.

Based on the given intersection pattern, we define some sets as follows.

Definition 5
Let the set P *be the set of numbers* Γ *such that* $v_\Gamma > 0$ *and let the set* Z *be the set of numbers* Γ *such that* $v_\Gamma = 0$, *i.e.,*

$$P = \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1 \text{ and } v_\Gamma > 0\},$$
$$Z = \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1 \text{ and } v_\Gamma = 0\}.$$

For any $0 \leq i \leq \lambda$, *let the set* P_i *be the set of numbers* Γ *such that the number of ones in the binary representation of* Γ *is* i *and* $v_\Gamma > 0$, *and let the set* Z_i *be the set of* Γ *such that the number of ones in the binary representation of* Γ *is* i *and* $v_\Gamma = 0$, *i.e.,*

$$P_i = \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, B(\Gamma) = i, \text{ and } v_\Gamma > 0\},$$
$$Z_i = \{\Gamma | 0 \leq \Gamma \leq 2^\lambda - 1, B(\Gamma) = i, \text{ and } v_\Gamma = 0\}. \qquad \square$$

From the definition of P and Z, we have the following obvious lemma, which gives a necessary condition on the existence of λ cubes to satisfy the given intersection pattern.

Lemma 1
If λ *cubes* $c_0, \ldots, c_{\lambda-1}$ *satisfy the given intersection pattern, then for any* $\Gamma \in P$, $C^\Gamma \neq 0$ *and for any* $\Gamma \in Z$, $C^\Gamma = 0$. $\square$

For any $\Gamma \in P$, we define a number k_Γ as follows.

Definition 6
For any $\Gamma \in P$, *define* $k_\Gamma = \log_2(v_\Gamma)$. $\square$

Since $v_\Gamma \in S = \{s | s = 0 \text{ or } s = 2^k, k = 0, 1, \ldots, n\}$, thus for any $\Gamma \in P$, k_Γ is an integer and $0 \leq k_\Gamma \leq n$.

For convenience, we represent a cube as a *cube-variable row vector* and a set of cubes as a *cube-variable matrix*. These are defined as follows.

Definition 7
Given a nonempty cube c *on* n *variables* $x_0, \ldots, x_{n-1}$, *we represent it by a* cube-variable row vector U *of length* n, *whose elements are from the set* $\{0, 1, *\}$. *If the* j*-th* ($0 \leq j \leq n-1$) *element* $U_j = 1$, *then the literal* x_j *appears in the cube* c*; if* $U_j = 0$, *then the literal* $\bar{x}_j$ *appears in the cube* c*; if* $U_j = *$, *then the cube* c *does not depend on the variable* x_j.

Given a set of λ *nonempty cubes* $c_0, \ldots, c_{\lambda-1}$ *on* n *variables* $x_0, \ldots, x_{n-1}$, *we represent them by a* cube-variable matrix D *of size* $\lambda \times n$, *so that the* i*-th row of the matrix is the cube-variable row vector of* c_i. $\square$

For example, a set of two cubes $c_0 = x_0\bar{x}_1$ and $c_1 = \bar{x}_0 x_2$ is represented as a cube-variable matrix

$$\begin{bmatrix} 1 & 0 & * \\ 0 & * & 1 \end{bmatrix}$$

Given a cube-variable row vector, the following simple lemma suggests how to obtain the number of minterms covered by the corresponding cube.

Lemma 2
If the cube-variable row vector of a nonempty cube contains k $*$'s, then the cube covers 2^k number of minterms. □

Definition 8
The negation of 0, 1 and $*$ are defined as 1, 0 and $*$, respectively. The negation of a cube-variable matrix (column vector) is the element-wise negation of the matrix (column vector). □

In what follows, we will say that a cube-variable matrix satisfies the given intersection pattern if the corresponding set of cubes satisfies the intersection pattern. The following lemma is straightforward.

Lemma 3
Suppose that a cube-variable matrix D satisfies the intersection pattern $(v_1, \ldots, v_{2^\lambda-1})$. Then D' satisfies the same intersection pattern if D' is obtained from D by column permutation or column negation. □

Before we go through the details of our proposed solution, we will briefly talk about the basic idea of our solution. Our solution is a column-based method: To synthesize a cube-variable matrix is equivalent to determine what each column of the matrix should be. Since each entry of the matrix is in the set $\{0, 1, *\}$, each column, which has λ entries, has totally 3^λ choices. Indeed, by the symmetry between different column choices and the disjoint relation among some cubes, we only need to consider a small subset of all 3^λ column choices as the candidate choices. Furthermore, by Lemma 3, since the order of the column does not matter, we only need to determine the number of occurrences of each candidate column choice in the cube-variable matrix, which we treat as unknowns. We establish a system of equations over those unknowns and the given intersection pattern. The λ-cube intersection problem can be solved by finding a non-negative solution to the system of equations.

3. A SPECIAL CASE OF THE λ-CUBE INTERSECTION PROBLEM

Here we consider a specific case in which $v_{2^\lambda-1} > 0$. First, we have the following theorem, which gives a necessary condition for λ cubes to satisfy the given intersection pattern.

Theorem 1
If $v_{2^\lambda-1} > 0$ and there exist λ cubes to satisfy the λ-cube intersection problem, then for any $0 \le \Gamma \le 2^\lambda - 1$, $\Gamma \in P$. □

PROOF. Based on Definition 3, for any $0 \le \Gamma \le 2^\lambda - 1$, $C^{2^\lambda-1} \subseteq C^\Gamma$. Therefore,

$$0 < v_{2^\lambda-1} = V(C^{2^\lambda-1}) \le V(C^\Gamma) = v_\Gamma.$$

By the definition of the set P, we have $\Gamma \in P$. □

In what follows, we will assume that there exist λ cubes to satisfy the given intersection pattern. Without loss of generality, we could assume that each entry of the cube-variable matrix is either 1 or $*$. Since $\prod_{i=0}^{\lambda-1} c_i \neq 0$, then for each column of the matrix D, it does not simultaneously contain both a 0 and a 1. Otherwise, $\prod_{i=0}^{\lambda-1} c_i = 0$. Therefore, each column of the matrix D contains either only 0's and $*$'s or only 1's and $*$'s. By Lemma 3, if we negate those columns of the matrix D that contain only 0's and $*$'s, then the new matrix D' obtained still satisfies the given intersection pattern. The matrix D' only contains 1's and $*$'s. Thus, we could assume that each column of the cube-variable matrix is in the set $\{1, *\}^\lambda$. The set $\{1, *\}^\lambda$ contains 2^λ elements. We denote those elements as $\psi_0, \psi_1, \ldots, \psi_{2^\lambda-1}$ with the help of the following definition.

Definition 9
Given any $0 \le \Gamma \le 2^\lambda - 1$, suppose that $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$, where $\gamma_i \in \{0, 1\}$. Define ψ_Γ to be a column vector of length λ with entries from the set $\{1, *\}$, such that the i-th element $(0 \le i \le \lambda - 1)$ of ψ_Γ is 1 if $\gamma_i = 0$ and is $*$ if $\gamma_i = 1$.
Define the set $\Psi = \{\psi_0, \psi_1, \ldots, \psi_{2^\lambda-1}\}$. □

For example, if $\lambda = 3$, then $\psi_0 = (1, 1, 1)^T$ [1] and $\psi_5 = (*, 1, *)^T$.
The basic idea of our proposed solution is to determine which column patterns from the set Ψ should be presented in the cube-variable matrix. Indeed, as pointed out in Section 2, we only need to determine how many column patterns of the form ψ_Γ are presented in the matrix. We define the number of occurrences of column pattern ψ_Γ as z_Γ.

Definition 10
For any $0 \le \Gamma \le 2^\lambda - 1$, define J_Γ to be the set of indices of the columns in the matrix D of the form ψ_Γ, i.e., $J_\Gamma = \{j | D_{\cdot j} = \psi_\Gamma\}$. Define z_Γ to be the cardinality of the set J_Γ. □

The following theorem gives relation between $\{z_0, \ldots z_{2^\lambda-1}\}$ and $\{k_0, \ldots, k_{2^\lambda-1}\}$.

Theorem 2
For any $0 \le L \le 2^\lambda - 1$, we have

$$k_L = \sum_{0 \le \Gamma \le 2^\lambda-1 : \Gamma \succeq L} z_\Gamma. \qquad (2)$$

□

PROOF. Since the total number of columns in matrix D is n, we have $\sum_{\Gamma=0}^{2^\lambda-1} z_\Gamma = n = k_0$, or $\sum_{0 \le \Gamma \le 2^\lambda-1 : \Gamma \succeq 0} z_\Gamma = k_0$. Thus, Equation (2) holds for $L = 0$.

Now consider $1 \le L \le 2^\lambda - 1$. Then L can be represented as $L = \sum_{j=0}^{r-1} 2^{l_j}$, where $1 \le r \le \lambda$ and $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$. Then, C^L represents the intersection of the set of cubes $c_{l_0}, \ldots, c_{l_{r-1}}$. The i-th entry in the cube-variable row vector of their intersection C^L is $*$ if and only if the column $D_{\cdot i}$ has $*$'s on the row $l_0, l_1, \ldots, l_{r-1}$. Therefore, the number of $*$'s in the cube-variable row vector of their intersection C^L is the number of columns in D, whose entries on the row $l_0, l_1, \ldots, l_{r-1}$ are all $*$'s, or

$$\sum_{\substack{0 \le \Gamma \le 2^\lambda-1: \\ (\psi_\Gamma)_{l_0} = \cdots = (\psi_\Gamma)_{l_{r-1}} = *}} z_\Gamma.$$

On the other hand,by Lemma 2, since $V(C^L) = 2^{k_L}$, the number of $*$'s in the cube-variable row vector of C^L is k_L. Therefore, we have

$$k_L = \sum_{\substack{0 \le \Gamma \le 2^\lambda-1: \\ (\psi_\Gamma)_{l_0} = \cdots = (\psi_\Gamma)_{l_{r-1}} = *}} z_\Gamma = \sum_{\substack{0 \le \Gamma \le 2^\lambda-1: \\ \gamma_{l_0} = \cdots = \gamma_{l_{r-1}} = 1}} z_\Gamma, \qquad (3)$$

where $L = \sum_{j=0}^{r-1} 2^{l_j}$ and $\Gamma = \sum_{i=0}^{\lambda-1} \gamma_i 2^i$.
By Definition 2, we can rewrite Equation (3) as

$$k_L = \sum_{0 \le \Gamma \le 2^\lambda-1 : \Gamma \succeq L} z_\Gamma.$$

□

[1] The superscript T here means the transpose of a matrix.

Note that Equation (2) is a linear equation in $z_0, \ldots, z_{2^\lambda-1}$ and holds for all $0 \leq L \leq 2^\lambda - 1$. Therefore, we can derive a system of 2^λ linear equations on unknowns $z_0, \ldots, z_{2^\lambda-1}$:

$$\sum_{0 \leq \Gamma \leq 2^\lambda - 1 : \Gamma \succeq L} z_\Gamma = k_L, \text{ for } L = 0, 1, \ldots, 2^\lambda - 1. \qquad (4)$$

We can represent the above system of linear equations in matrix form, as shown by the following theorem.

Theorem 3
Let vector $\vec{k} = (k_0, \ldots, k_{2^\lambda-1})^T$ and vector $\vec{z} = (z_0, \ldots, z_{2^\lambda-1})^T$. Then we can represent the system of 2^λ linear equations (4) in matrix form as

$$R_\lambda \vec{z} = \vec{k}, \qquad (5)$$

where R_λ is a $2^\lambda \times 2^\lambda$ square matrix recursively defined as follows:

$$R_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, R_i = \begin{bmatrix} R_{i-1} & R_{i-1} \\ 0 & R_{i-1} \end{bmatrix}, \text{ for } i = 2, \ldots, \lambda. \quad \square$$

Due to space constraints, we omit the proof.

It is not hard to see that $\det(R_\lambda) = 1$. Therefore, R_λ is invertible. The following theorem shows what R_λ^{-1} is.

Theorem 4
R_λ^{-1} is recursively defined as follows:

$$R_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, R_i^{-1} = \begin{bmatrix} R_{i-1}^{-1} & -R_{i-1}^{-1} \\ 0 & R_{i-1}^{-1} \end{bmatrix}, \text{ for } i = 2, \ldots, \lambda. \quad \square$$

Therefore, given $k_0, k_1, \ldots, k_{2^\lambda-1}$, we can get $z_0, z_1, \ldots, z_{2^\lambda-1}$ as $\vec{z} = R_\lambda^{-1}\vec{k}$.

Since for any $0 \leq \Gamma \leq 2^\lambda - 1$, z_Γ is the cardinality of the set J_Γ, therefore, z_Γ must be a non-negative integer. By Theorem 4, R_λ^{-1} is an integer matrix. Therefore, $z_0, \ldots, z_{2^\lambda-1}$ are always integers. Thus, a necessary condition for the existence of λ cubes to satisfy the given intersection pattern is that the vector $R_\lambda^{-1}\vec{k}$ has all entries non-negative. On the other hand, from Equation (5), we can see that the intersection pattern $(2^{k_1}, \ldots, 2^{k_{2^\lambda-1}})$ only depends on $z_0, \ldots, z_{2^\lambda-1}$. Therefore, as long as the vector $R_\lambda^{-1}\vec{k}$ has all entries non-negative, there exist λ cubes to satisfy the given intersection pattern. In summary, we have the following corollary.

Corollary 1
The necessary and sufficient condition for the existence of λ cubes to satisfy the given intersection pattern is that the vector $R_\lambda^{-1}\vec{k}$ has all entries non-negative, where $\vec{k} = (k_0, k_1, \ldots, k_{2^\lambda-1})^T$ and R_λ^{-1} is defined in Theorem 4. $\square$

Example 3
Given $v_1 = 4$, $v_2 = 4$, and $v_3 = 1$, determine whether there exists a set of 2 cubes c_0 and c_1 on 4 variables to satisfy the intersection pattern (v_1, v_2, v_3).
Solution: *From the given conditions, we have $\vec{k} = (4, 2, 2, 0)^T$. Since*

$$R_2^{-1} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then by Equation (5), we get $\vec{z} = (0, 2, 2, 0)^T$. Therefore, there are two ψ_1's and two ψ_2's in the cube-variable matrix of c_0 and c_1. One realization of the cube-variable matrix is

$$\begin{bmatrix} * & * & 1 & 1 \\ 1 & 1 & * & * \end{bmatrix}$$

and the corresponding cubes are $c_0 = x_2x_3$ and $c_1 = x_0x_1$. $\square$

4. GENERAL λ-CUBE INTERSECTION PROBLEM

In this section, we consider the more general situation where $v_{2^\lambda-1} \geq 0$.

4.1 Necessary Conditions on the Positive v_Γ's

We first have the following theorem applicable for numbers $v_\Gamma > 0$.

Theorem 5
If there exist λ cubes $c_0, \ldots, c_{\lambda-1}$ to satisfy the intersection pattern, then for any $1 \leq L \leq 2^\lambda - 1$ such that $v_L > 0$, we have that for any $1 \leq \Gamma \leq 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$. $\square$

PROOF. For any $1 \leq \Gamma \leq 2^\lambda - 1$ such that $L \succeq \Gamma$, it is not hard to see that $C^L \subseteq C^\Gamma$. Therefore, $0 < v_L = V(C^L) \leq V(C^\Gamma) = v_\Gamma$. $\square$

If a set of cubes is pairwise non-disjoint, then it has the following property.

Lemma 4
If a set of r cubes $c_{l_0}, \ldots, c_{l_{r-1}}$ $(3 \leq r \leq \lambda, 0 \leq l_0 < \cdots < l_{r-1} \leq \lambda - 1)$ is pairwise non-disjoint, i.e., for any $0 \leq i < j \leq r-1$, $c_{l_i} \cdot c_{l_j} \neq 0$, then their intersection $\prod_{i=0}^{r-1} c_{l_i}$ is nonempty. $\square$

PROOF. By contraposition, suppose that $\prod_{i=0}^{r-1} c_{l_i} = 0$. Consider the cube-variable matrix on these r cubes. Since their intersection is empty, there exists a column in the matrix that contains both a 0 and a 1. The cube corresponding to the 0 entry and the cube corresponding to the 1 entry are disjoint. This contradicts the assumption that the given set of cubes is pairwise non-disjoint. $\square$

Alternatively, Lemma 4 can be stated on the numbers v_Γ. This gives a necessary condition for the existence of a set of cubes to satisfy the given intersection pattern.

Theorem 6
Suppose that there exist λ cubes $c_0, \ldots, c_{\lambda-1}$ to satisfy the given intersection pattern. If a set of r $(3 \leq r \leq \lambda)$ numbers $0 \leq l_0 < \cdots < l_{r-1} \leq \lambda - 1$ satisfies that for any $0 \leq i < j \leq r-1$, $v_{(2^{l_i}+2^{l_j})} > 0$, then for $L = \sum_{i=0}^{r-1} 2^{l_i}$, $v_L > 0$. $\square$

For example, suppose that in a 4-cube intersection problem we are given $v_3 > 0$, $v_9 > 0$, and $v_{10} > 0$. If there exist 4 cubes to satisfy the given intersection pattern, then since $V(c_0c_1) > 0$, $V(c_0c_3) > 0$, and $V(c_1c_3) > 0$, we must have $v_{11} = V(c_0c_1c_3) > 0$.

If both the conditions in Theorem 5 and 6 are satisfied, then we have the following theorem, which will play an important role in proving the necessary and sufficient condition later.

Theorem 7
Suppose that the given intersection pattern satisfies that

1. *For any $1 \leq L \leq 2^\lambda - 1$, if $v_L > 0$, then for any $1 \leq \Gamma \leq 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$.*

2. *For any set of r $(3 \leq r \leq \lambda)$ numbers $0 \leq l_0 < \cdots < l_{r-1} \leq \lambda - 1$, if it satisfies that for any $0 \leq i < j \leq r-1$, $v_{(2^{l_i}+2^{l_j})} > 0$, then for the number $L = \sum_{i=0}^{r-1} 2^{l_i}$, $v_L > 0$.*

Then, a necessary and sufficient condition for a set of λ nonempty cubes to satisfy the condition that for any $\Gamma \in P$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z$, $C^\Gamma = 0$ is that for any $\Gamma \in P_2$, $C^\Gamma \neq 0$ and for any $\Gamma \in Z_2$, $C^\Gamma = 0$. $\square$

Due to space constraints, we omit the proof.

4.2 Compatible Column Pattern Set

In the general case, the cube-variable matrix consists of 0, 1 and $*$ and so does each column of the matrix. There are totally 3^λ different choices of patterns for each column. However, not all combinations of 0, 1 and $*$ as a column vector can be presented in the matrix. For example, if the given intersection pattern indicates that $c_i \cdot c_j \neq 0$, then those column patterns that have a 0 on the i-th entry and a 1 on the j-th entry cannot be presented in the matrix. On the other hand, some kinds of column patterns must be presented at least once in the matrix. For example, if the given intersection pattern indicates that $c_i \cdot c_j = 0$, then at least one of the column patterns that have a 0 on the i-th entry and a 1 on the j-th entry or have a 1 on the i-th entry and a 0 on the j-th entry must be presented in the matrix. In this section, we will show what kind of column patterns can be presented in the matrix. For this purpose, we first introduce the *compatible column pattern set* for numbers $\Gamma \in Z_2$.

Definition 11
Suppose that $\Gamma \in Z_2$ and $\Gamma = 2^i + 2^j$, where $0 \leq i < j \leq \lambda - 1$. The compatible column pattern set for Γ is the set of column vectors W of length λ with entries from the set $\{0, 1, *\}$, such that

1. *$W_i = 0$ and $W_j = 1$ or $W_i = 1$ and $W_j = 0$,*

2. *for any number $L \in P_2$ such that $L = 2^k + 2^l$, where $0 \leq k < l \leq \lambda - 1$, the situation that $W_k = 0$ and $W_l = 1$ or $W_k = 1$ and $W_l = 0$ does not happen.* □

It is not hard to see that if a cube-variable column vector is in the compatible column pattern set for a $\Gamma \in Z_2$, then the negation of that cube-variable column vector is also in that set. Therefore, we define the *representative compatible column pattern set* as follows.

Definition 12
The representative compatible column pattern set ρ_Γ for $\Gamma \in Z_2$ is a subset of the compatible column pattern set for Γ such that the first non-$*$ entry of each element in the representative set is 0. □

Example 4
Consider a 4-cube intersection problem with

$$P_2 = \{(0011)_2, (0101)_2, (1001)_2\},$$
$$Z_2 = \{(0110)_2, (1010)_2, (1100)_2\}.$$

The compatible column pattern set for $\Gamma = (0110)_2 \in Z_2$ is

$$\{(*010)^T, (*101)^T, (*011)^T, (*100)^T, (*01*)^T, (*10*)^T\}.$$

The representative compatible column pattern set for $\Gamma = (0110)_2$ is $\{(*010)^T, (*011)^T, (*01*)^T\}$. □

Definition 13
We define the set Y as the union of the representative compatible column pattern sets ρ_Γ for all $\Gamma \in Z_2$, i.e., $Y = \bigcup_{\Gamma \in Z_2} \rho_\Gamma$. We define the set $F = Y \cup \Psi$. □

The following lemma shows that only those column patterns in the set F are needed to construct the cube-variable matrix.

Lemma 5
If there exists a cube-variable matrix D to satisfy the given intersection pattern, then there exists another matrix D' which also satisfies the given intersection pattern and each column of which is in the set F. □

PROOF. First, we argue that for any column of D which contains both a 0 and a 1 entry, the column is in the compatible column pattern set of a certain $\Gamma \in Z_2$. In fact, if a column r $(0 \leq r \leq n-1)$ of D has the i-th entry being 0 and the j-th entry being 1, where $0 \leq i, j \leq \lambda - 1$ and $i \neq j$, then it is not hard to show that the column is in the compatible column pattern set for the number $(2^i + 2^j) \in Z_2$.

We can construct a D' from D as follows. For any column $0 \leq r \leq \lambda - 1$:

1. If $D_{\cdot r}$ contains only 1's and $*$'s, we let $D'_{\cdot r}$ be $D_{\cdot r}$. Then $D'_{\cdot r}$ is in the set Ψ.

2. If $D_{\cdot r}$ contains only 0's and $*$'s, we let $D'_{\cdot r}$ be the negation of the column $D_{\cdot r}$. Then $D'_{\cdot r}$ is in the set Ψ.

3. If $D_{\cdot r}$ contains both a 0 and a 1 and the first non-$*$ entry of $D_{\cdot r}$ is 0, we let $D'_{\cdot r}$ be $D_{\cdot r}$. Then, there exists a $\Gamma \in Z_2$ such that $D'_{\cdot r}$ is in the set ρ_Γ.

4. If $D_{\cdot r}$ contains both a 0 and a 1 and the first non-$*$ entry of $D_{\cdot r}$ is 1, we let $D'_{\cdot r}$ be the negation of the column $D_{\cdot r}$. Then, there exists a $\Gamma \in Z_2$ such that $D'_{\cdot r}$ is in the set ρ_Γ.

Then, by the above construction, each column of D' is in the set F. Further, D' is obtained from D by column negations. Thus, by Lemma 3, D' also satisfies the given intersection pattern. □

Based on Lemma 5, we only need to answer whether there exists a cube-variable matrix with columns from the set F to satisfy the given intersection pattern. The following lemma states that if such a matrix exists, then for each $\Gamma \in Z_2$, at least one of the column pattern elements from the set ρ_Γ must be presented in that matrix.

Lemma 6
If a cube-variable matrix D with columns from the set F satisfies the given intersection pattern, then for any $\Gamma \in Z_2$, there exists a column in D which is in the set ρ_Γ. □

PROOF. For any $\Gamma \in Z_2$, suppose that $\Gamma = 2^i + 2^j$, where $0 \leq i < j \leq \lambda - 1$. Since the cube-variable matrix satisfies the given intersection pattern, then based on Lemma 1, for the $\Gamma \in Z_2$, we must have $C^\Gamma = 0$ or $c_i \cdot c_j = 0$. Thus, there must exist a column r in D, such that $D_{ir} = 0$ and $D_{jr} = 1$ or $D_{ir} = 1$ and $D_{jr} = 0$. Now consider any $L \in P_2$. Suppose that $L = 2^k + 2^l$, where $0 \leq k < l \leq \lambda - 1$. Since the necessary condition for the cube-variable matrix to satisfy a given intersection pattern is that for the $L \in P_2$, $C^L \neq 0$, the situation that $D_{kr} = 0$ and $D_{lr} = 1$ or $D_{kr} = 1$ and $D_{lr} = 0$ cannot happen. Therefore, the column r of D is in the compatible column pattern set for Γ. Further, since all the columns of D are in the set F, then column r must be in the set ρ_Γ. □

4.3 A Necessary and Sufficient Condition

In this section, we will show a necessary and sufficient condition for the existence of a set of cubes to satisfy the given intersection pattern. As a byproduct, the proof provides a way of synthesizing a set of cubes to satisfy the given intersection pattern. Based on Lemma 5, we only need to consider cube-variable matrix that consists of column patterns from the set F. The basic idea to solve the general case problem is similar to that applied in the special case — we will establish relations between the numbers of occurrences of those elements of the set F in the cube-variable matrix and the k_Γ's. First, we define *root cube-variable matrix*, which links the general case problem to the special case problem we discussed in Section 3.

Definition 14
Given a cube-variable matrix D on λ cubes $c_0, \ldots, c_{\lambda-1}$, we define root cube-variable matrix *$t(D)$ of D as the cube-variable matrix formed by replacing the 0 entries in D with 1's and keeping the other entries in D unchanged. The set of cubes $c'_0, \ldots, c'_{\lambda-1}$ corresponding to the root matrix is called the set of* root cubes *to the original set of cubes.* □

For example, the root matrix of the cube-variable matrix

$$\begin{bmatrix} 1 & 0 & * \\ 0 & * & 1 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & * \\ 1 & * & 1 \end{bmatrix}.$$

The set of root cubes is $c'_0 = x_0 x_1$ and $c'_1 = x_0 x_2$.

Based on the definition of the set of root cubes, it is not hard to prove the following lemma.

Lemma 7
Suppose that the set of root cubes to the set of original cubes $c_0, \ldots, c_{\lambda-1}$ is $c'_0, \ldots, c'_{\lambda-1}$. Then, for any $\Gamma \in P$, we have $V(C'^{\Gamma}) = V(C^{\Gamma})$. $\square$

Since the root matrix $t(D)$ is a matrix containing only 1's and $*$'s, we can apply the definition of z_Γ in Definition 10 to $t(D)$. Then, based on the fact that for any $\Gamma \in P$, $V(C'^{\Gamma}) = V(C^{\Gamma}) = 2^{k_\Gamma}$, it is not hard to show that the following theorem characterizing the relation between z_Γ's and k_L's holds.

Theorem 8
If there exist λ cubes to satisfy the given intersection pattern, then for any $L \in P$, $\sum_{0 \le \Gamma \le 2^\lambda - 1 : \Gamma \succeq L} z_\Gamma = k_L$, where z_Γ's are defined on the root matrix $t(D)$ according to Definition 10. $\square$

By the similar definition of root cube-variable matrix, we define *root column vector* as follows.

Definition 15
Given a column vector W with each element in the set $\{0, 1, *\}$, define its root column vector *$t(W)$ as the column vector obtained from W by replacing the 0 entries in W with 1's and keeping the other entries in W unchanged.* $\square$

Based on the definition of the root column vector, we can regroup the elements in the set Y according to their root column vectors, which results to the following definition. The relation between the elements in the set Y and their root column vectors will be used later to derive a set of inequalities on the numbers of occurrences of the elements of the set F in the cube-variable matrix (See Theorem 9).

Definition 16
We define the set M to be the set of numbers $0 \le \Gamma \le 2^\lambda - 1$ such that there exists an element in the set Y, whose root column vector is ψ_Γ, i.e.,

$$M = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1, \text{ s.t. } \exists W \in Y \text{ s.t. } t(W) = \psi_\Gamma\}.$$

Define $\overline{M}$ as $\overline{M} = \{\Gamma | 0 \le \Gamma \le 2^\lambda - 1, \Gamma \notin M\}$.

For any $\Gamma \in M$, we define the set Y_Γ to be the set of elements in the set Y such that their root column vectors are ψ_Γ, i.e., $Y_\Gamma = \{W | W \in Y \text{ and } t(W) = \psi_\Gamma\}$. $\square$

Notice that the sets Y_Γ $(\Gamma \in M)$ form a partition of the set Y.

Example 5
For the intersection pattern shown in Example 4, we have $Z_2 = \{6, 10, 12\}$ and

$$\begin{aligned} \rho_6 &= \{(*010)^T, (*011)^T, (*01*)^T\}, \\ \rho_{10} &= \{(*001)^T, (*011)^T, (*0*1)^T\}, \\ \rho_{12} &= \{(*010)^T, (*001)^T, (**01)^T\}. \end{aligned}$$

Thus,

$$Y = \{(*010)^T, (*001)^T, (*011)^T, (**01)^T, (*0*1)^T, (*01*)^T\},$$
$$M = \{1, 3, 5, 9\},$$

and $Y_1 = \{(*010)^T, (*001)^T, (*011)^T\}$, $Y_3 = \{(**01)^T\}$, $Y_5 = \{(*0*1)^T\}$, and $Y_9 = \{(*01*)^T\}$. $\square$

Based on Lemma 5, we could assume that each column of the cube-variable matrix is from the set $F = Y \cup \Psi$. To solve the general case problem, we only need to determine the number of occurrences of each element of the set F in the cube-variable matrix. In order to establish equations, we first define the number of occurrences of each element of the set Y in the cube-variable matrix, which is actually defined on each partition Y_Γ of Y, as stated by the following definition.

Definition 17
For any $\Gamma \in M$, we let the $|Y_\Gamma|$ elements in the set Y_Γ be $\delta_{\Gamma,0}, \ldots, \delta_{\Gamma,|Y_\Gamma|-1}$. For any $0 \le i \le |Y_\Gamma| - 1$, we define $K_{\Gamma,i}$ to be the set of indices of the columns in the matrix D of the form $\delta_{\Gamma,i}$, i.e., $K_{\Gamma,i} = \{k | D_{\cdot k} = \delta_{\Gamma,i}\}$. We define $w_{\Gamma,i}$ to be the cardinality of the set $K_{\Gamma,i}$. $\square$

The following theorem establishes a set of linear inequalities on $w_{\Gamma,i}$'s and z_Γ's, where z_Γ's are defined on the root matrix according to Definition 10.

Theorem 9
Suppose that there exists a cube-variable matrix D to satisfy the given intersection pattern, whose columns are from the set F. Then, we have that for any $\Gamma \in M$,

$$\sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \le z_\Gamma, \tag{6}$$

where z_Γ's are defined on the root matrix $t(D)$ according to Definition 10. We also have that for any $L \in Z_2$,

$$\sum_{\substack{\Gamma \in M, 0 \le i \le |Y_\Gamma|-1: \\ \delta_{\Gamma,i} \in \rho_L}} w_{\Gamma,i} \ge 1. \tag{7}$$

$\square$

PROOF. Consider any $\Gamma \in M$. For any number $k \in \bigcup_{i=0}^{|Y_\Gamma|-1} K_{\Gamma,i}$, the column vector $D_{\cdot k}$ is in the set Y_Γ. Thus, the root column vector of $D_{\cdot k}$ is ψ_Γ. Thus, $k \in J_\Gamma$, where J_Γ is defined on the root matrix $t(D)$. Therefore, $\bigcup_{i=0}^{|Y_\Gamma|-1} K_{\Gamma,i} \subseteq J_\Gamma$. As a result, $\left| \bigcup_{i=0}^{|Y_\Gamma|-1} K_{\Gamma,i} \right| \le |J_\Gamma|$, or $\sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \le z_\Gamma$.

By Lemma 6, for any $L \in Z_2$, there exists a column in D which is in the set ρ_L. Suppose that column is of the form $\delta_{\Gamma^*,i^*} \in \rho_L$, where $\Gamma^* \in M$ and $0 \le i \le |Y_{\Gamma^*}| - 1$. Thus,

$$1 \le w_{\Gamma^*,i^*} \le \sum_{\substack{\Gamma \in M, 0 \le i \le |Y_\Gamma|-1: \\ \delta_{\Gamma,i} \in \rho_L}} w_{\Gamma,i}.$$

$\square$

Example 6
For the intersection pattern given in Example 4, based on the result shown in Example 5, we have

$$\begin{aligned} &\delta_{1,0} = (*010)^T, \delta_{1,1} = (*001)^T, \delta_{1,2} = (*011)^T, \\ &\delta_{3,0} = (**01)^T, \delta_{5,0} = (*0*1)^T, \delta_{9,0} = (*01*)^T. \end{aligned}$$

The set of equations (6) for all $\Gamma \in M$ in this example is

$$\begin{cases} w_{\Gamma,0} \le z_\Gamma, \text{ for any } \Gamma \in \{3, 5, 9\} \\ w_{1,0} + w_{1,1} + w_{1,2} \le z_1 \end{cases}$$

The set of equations (7) for all $L \in Z_2$ in this example is

$$\begin{cases} w_{1,0} + w_{1,2} + w_{9,0} \ge 1 \\ w_{1,1} + w_{1,2} + w_{5,0} \ge 1 \\ w_{1,0} + w_{1,1} + w_{3,0} \ge 1 \end{cases} \quad \square$$

Finally, combining the conditions of Theorem 5, 6, 8, and 9, we can derive the following necessary and sufficient condition.

Theorem 10
There exists a cube-variable matrix D to satisfy the given intersection pattern $(v_1, \ldots, v_{2^\lambda - 1})$ if and only if

1. *for any $1 \le L \le 2^\lambda - 1$, if $v_L > 0$, then for any $1 \le \Gamma \le 2^\lambda - 1$ such that $L \succeq \Gamma$, $v_\Gamma > 0$,*

2. *for any set of r $(3 \le r \le \lambda)$ numbers $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$, if it satisfies that for any $0 \le i < j \le r - 1$, $v_{(2^{l_i} + 2^{l_j})} > 0$, then for the number $L = \sum_{i=0}^{r-1} 2^{l_i}$, $v_L > 0$,*

3. *the system of equations on unknowns* $\tilde{z}_\Gamma$ (*for all* $0 \le \Gamma \le 2^\lambda - 1$) *and* $\tilde{w}_{\Gamma,i}$ (*for all* $\Gamma \in M$ *and* $0 \le i \le |Y_\Gamma| - 1$)

$$\sum_{0 \le \Gamma \le 2^\lambda - 1 : \Gamma \succeq L} \tilde{z}_\Gamma = k_L, \text{ for all } L \in P$$
$$\sum_{i=0}^{|Y_\Gamma|-1} \tilde{w}_{\Gamma,i} \le \tilde{z}_\Gamma, \text{ for all } \Gamma \in M \tag{8}$$
$$\sum_{\substack{\Gamma \in M, 0 \le i \le |Y_\Gamma|-1: \\ \delta_{\Gamma,i} \in \rho_L}} \tilde{w}_{\Gamma,i} \ge 1, \text{ for all } L \in Z_2$$

has a non-negative integer solution. □

PROOF. "**only if**" part: Statement 1 in the theorem is due to Theorem 5 and Statement 2 in the theorem is due to Theorem 6.

Since D satisfies the given intersection pattern, then by Lemma 5, there exists another matrix D' which also satisfies the given intersection pattern and each column of which is in the set F. For any $0 \le \Gamma \le 2^\lambda - 1$, let $\tilde{z}_\Gamma = z_\Gamma$, where z_Γ's are defined on the root matrix $t(D')$ according to Definition 10. For any $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$, let $\tilde{w}_{\Gamma,i} = w_{\Gamma,i}$, where $w_{\Gamma,i}$'s are defined on the matrix D' according to Definition 17. By Theorem 8 and 9, the set of numbers $\tilde{z}_\Gamma$ and $\tilde{w}_{\Gamma,i}$ satisfies the system of equations (8). Since $\tilde{z}_\Gamma$ is the cardinality of the set J_Γ and $\tilde{w}_{\Gamma,i}$ is the cardinality of the set $K_{\Gamma,i}$, therefore, $\tilde{z}_\Gamma$'s and $\tilde{w}_{\Gamma,i}$'s are all non-negative integers. Thus, the system of equations (8) has a non-negative solution.

"**if**" part: Let a non-negative solution to the system of equations (8) be $\tilde{z}_\Gamma = z_\Gamma$, for all $0 \le \Gamma \le 2^\lambda - 1$, and $\tilde{w}_{\Gamma,i} = w_{\Gamma,i}$, for all $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$. Since for all $0 \le \Gamma \le 2^\lambda - 1$, $z_\Gamma \ge 0$, for all $\Gamma \in M$ and $0 \le i \le |Y_\Gamma| - 1$, $w_{\Gamma,i} \ge 0$, and for all $\Gamma \in M$, $\sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \le z_\Gamma$, then, we can construct a cube-variable matrix D so that

1. for all $\Gamma \in \overline{M}$, the matrix contains z_Γ columns of the form ψ_Γ,
2. for all $\Gamma \in M$, the matrix contains $z_\Gamma - \sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i}$ columns of the form ψ_Γ, and
3. for all $\Gamma \in M$ and all $0 \le i \le |Y_\Gamma| - 1$, the matrix contains $w_{\Gamma,i}$ columns of the form $\delta_{\Gamma,i}$.

All columns of the matrix D are in the set F. Next, we prove that the matrix D satisfies the given intersection pattern.

For any $L \in Z_2$, suppose $L = 2^i + 2^j$, where $0 \le i < j \le \lambda - 1$. Since $\sum_{\substack{\Gamma \in M, 0 \le k \le |Y_\Gamma|-1: \\ \delta_{\Gamma,k} \in \rho_L}} w_{\Gamma,k} \ge 1$, there exists a $\Gamma^* \in M$ and a $0 \le k^* \le |Y_{\Gamma^*}| - 1$, such that $\delta_{\Gamma^*,k^*} \in \rho_L$ and $w_{\Gamma^*,k^*} \ge 1$. Therefore, the matrix D contains a column from the set ρ_L. Based on the definition of ρ_L, $C^L = c_i \cdot c_j = 0$. Thus, for any $L \in Z_2$, $C^L = 0$.

Now consider any $L \in P_2$. Suppose $L = 2^i + 2^j$, where $0 \le i < j \le \lambda - 1$. We argue that $C^L = c_i \cdot c_j \ne 0$. Otherwise, $c_i \cdot c_j = 0$. Therefore, there exists a column r in D, such that $D_{ir} = 0$ and $D_{jr} = 1$ or $D_{ir} = 1$ and $D_{jr} = 0$. Since all the columns of D are in the set F, thus the column $D_{\cdot r}$ must be in the set Y. However, based on the definition of representative compatible column pattern set, each element W in the set Y satisfies that for the $L \in P_2$, the situation that $W_i = 0$ and $W_j = 1$ or $W_i = 1$ and $W_j = 0$ does not happen. Therefore, the column $D_{\cdot r}$ does not belong to the set Y. We get a contradiction. Thus, for any $L \in P_2$, we have $C^L \ne 0$.

Since the given intersection pattern satisfies the conditions of Theorem 7, then, based on Theorem 7, we have that for any $\Gamma \in Z$, $C^\Gamma = 0$ and for any $\Gamma \in P$, $C^\Gamma \ne 0$. Thus, for all these $\Gamma \in Z$, $V(C^\Gamma) = v_\Gamma = 0$.

Now consider any $L \in P$. When $L = 0$, it is not hard to see that the total number of columns in D is n.

For any $L \in P$ and $L > 0$, L can be represented as $L = \sum_{j=0}^{r-1} 2^{l_j}$, where $1 \le r \le \lambda$ and $0 \le l_0 < \cdots < l_{r-1} \le \lambda - 1$. Since $C^L \ne 0$, the number of $*$'s in the cube-variable row vector C^L is the number of columns in D, whose entries on the row $l_0, l_1, \ldots, l_{r-1}$ are all $*$'s. Note that for any $0 \le \Gamma \le 2^\lambda - 1$, the column pattern ψ_Γ has all entries on the row $l_0, l_1, \ldots, l_{r-1}$ being $*$'s if and only if $\Gamma \succeq L$. Since the root column vector of $\delta_{\Gamma,i}$ is ψ_Γ, thus for any $\Gamma \in M$ and any $0 \le i \le |Y_\Gamma| - 1$, the column pattern $\delta_{\Gamma,i}$ has all entries on the row $l_0, l_1, \ldots, l_{r-1}$ being $*$'s if and only if $\Gamma \succeq L$. Therefore, the number of columns in D, whose entries on the row $l_0, l_1, \ldots, l_{r-1}$ are all $*$'s, is

$$\sum_{\substack{\Gamma \in \overline{M}: \\ \Gamma \succeq L}} z_\Gamma + \sum_{\substack{\Gamma \in M: \\ \Gamma \succeq L}} \left(z_\Gamma - \sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i} \right) + \sum_{\substack{\Gamma \in M: \\ \Gamma \succeq L}} \sum_{i=0}^{|Y_\Gamma|-1} w_{\Gamma,i}$$
$$= \sum_{0 \le \Gamma \le 2^\lambda - 1 : \Gamma \succeq L} z_\Gamma = k_L.$$

Therefore, the number of $*$'s in the row vector C^L is k_L. Since $C^L \ne 0$, by Lemma 2, $V(C^L) = 2^{k_L}$. Thus, for any $L \in P$ and $L > 0$, $V(C^L) = 2^{k_L} = v_L$.

In summary, the matrix D has n columns and for any $1 \le \Gamma \le 2^\lambda - 1$, $V(C^\Gamma) = v_\Gamma$. Thus, the matrix D satisfies the given intersection pattern. □

Comment: The above proof provides a way of synthesizing a cube-variable matrix to satisfy the given intersection pattern when the three conditions are all satisfied.

Example 7
In a 3*-cube intersection problem on* 4 *variables* $x_0, \ldots, x_3$, *suppose that the intersection pattern is given as*

$$v_1 = 4, v_2 = 4, v_3 = 0, v_4 = 4, v_5 = 1, v_6 = 2, v_7 = 0.$$

First, it is not hard to check that both Statement 1 and Statement 2 in Theorem 10 hold for the given pattern.

By convention, $v_0 = 2^4 = 16$. *Therefore, we have*

$$P = \{0, 1, 2, 4, 5, 6\}, \qquad Z = \{3, 7\},$$
$$k_0 = 4, k_1 = 2, k_2 = 2, k_4 = 2, k_5 = 0, k_6 = 1.$$

For the given intersection pattern, we have $Z_2 = \{3\}$ *and* $\rho_3 = \{(01*)^T\}$.

Thus, $Y = \{(01*)^T\}, M = \{4\}$ *and* $Y_4 = \{(01*)^T\}$. *Thus,* $\delta_{4,0} = (01*)^T$.

The system of equations (8) *in this example is*

$$\begin{aligned} &\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \tilde{z}_3 + \tilde{z}_3 + \tilde{z}_4 + \tilde{z}_6 + \tilde{z}_7 = 4, \\ &\tilde{z}_1 + \tilde{z}_3 + \tilde{z}_5 + \tilde{z}_7 = 2, \quad \tilde{z}_2 + \tilde{z}_3 + \tilde{z}_6 + \tilde{z}_7 = 2, \\ &\tilde{z}_4 + \tilde{z}_5 + \tilde{z}_6 + \tilde{z}_7 = 2, \quad \tilde{z}_5 + \tilde{z}_7 = 0, \quad \tilde{z}_6 + \tilde{z}_7 = 1. \\ &\tilde{w}_{4,0} \le \tilde{z}_4, \quad \tilde{w}_{4,0} \ge 1 \end{aligned} \tag{9}$$

The above system of equations (9) *has a non-negative solution*

$$\tilde{z}_1 = \tilde{z}_3 = \tilde{z}_4 = \tilde{z}_6 = 1, \tilde{z}_0 = \tilde{z}_2 = \tilde{z}_5 = \tilde{z}_7 = 0, \tilde{w}_{4,0} = 1.$$

Thus, Statement 3 in Theorem 10 also holds. Therefore, there exists a cube-variable matrix to satisfy the given intersection pattern. Based on the proof of Theorem 10, we can synthesize a cube-variable matrix that satisfies the given intersection pattern based on the above non-negative solution as

$$\begin{bmatrix} * & * & 0 & 1 \\ 1 & * & 1 & * \\ 1 & 1 & * & * \end{bmatrix}$$

and the corresponding cubes are $c_0 = \bar{x}_2 x_3$, $c_1 = x_0 x_2$, *and* $c_2 = x_0 x_1$. *It is not hard to verify that the set of cubes* c_0, c_1, c_2 *satisfies the given intersection pattern.* □

As shown by Theorem 10, a critical step in solving the λ-cube intersection problem is to find a non-negative solution to the system of equations (8). The following theorem shows that to find a non-negative solution to the system of equations (8) is equivalent to find a non-negative solution to an alternative system of equations with fewer unknowns and equations.

Theorem 11
The system of equations (8) *has a non-negative integer solution if and only if the system of equations on unknowns* $\hat{z}_\Gamma$ (*for all* $\Gamma \in \overline{M}$) *and* $\hat{w}_{\Gamma,i}$ (*for all* $\Gamma \in M$ *and* $0 \le i \le |Y_\Gamma| - 1$)

$$\begin{aligned}\sum_{\Gamma \in \overline{M}, \Gamma \succeq L} \hat{z}_\Gamma + \sum_{\Gamma \in M, \Gamma \succeq L} \sum_{i=0}^{|Y_\Gamma|-1} \hat{w}_{\Gamma,i} = k_L, \text{ for all } L \in P \\ \sum_{\substack{\Gamma \in M, 0 \le i \le |Y_\Gamma|-1: \\ \delta_{\Gamma,i} \in \rho_L}} \hat{w}_{\Gamma,i} \ge 1, \text{ for all } L \in Z_2\end{aligned} \tag{10}$$

has a non-negative integer solution. □

Due to space constraints, we omit the proof here.

Note that the system of equations (10) has $|M|$ fewer unknowns and $|M|$ fewer inequalities than the original system of equations (8). Thus, a certain amount of computation will be saved by solving the alternative system of equations (10).

5. EXPERIMENTAL RESULTS

We test our algorithm on two-level logic benchmarks that accompany the two-level logic minimizer Espresso [5]. For each benchmark, we ignore the output part of the cubes and just set the number of outputs to be one. We optimize each modified benchmark by Espresso and then call a program to generate an intersection pattern file of that benchmark. This intersection pattern file serves as the input to our program.

We perform two sets of experiments to test our algorithm. In the first set of experiments, we test our algorithm on solving special case problems. The major goal is to study the runtime of our algorithm to solve special case problems. The benchmarks we tested are listed in Table 1. Since just a few benchmarks generate a special intersection pattern. We manually create some test cases. For example, the benchmark `mark1_11` is created from the original benchmark `mark1` by deleting five cubes. Notice that by deleting some cubes, the new benchmark still has its intersection of all cubes nonempty. Not surprisingly, the runtime increases exponentially with the number of cubes λ. This is because the number of unknowns increases exponentially with λ. However, since the size of the inputs to our program is $O(2^\lambda)$, which is proportional to the number of intersections, the runtime complexity compared to the size of the inputs is linear. Further, for the benchmark `shift`, although the number of unknowns is more than 2 million, our algorithm is able to obtain the solution in about 70 seconds.

Table 1: Number of unknowns and runtime for special case problems.

circuit	#cubes	#inputs	#unknowns	time (s)
`newtpla2`	9	10	512	0
`in3`	10	35	1024	0
`mark1_11`	11	20	2048	0.01
`mark1_12`	12	20	4096	0.04
`mark1_13`	13	20	8192	0.08
`mark1_14`	14	20	16384	0.2
`mark1_15`	15	20	32768	0.48
`mark1`	16	20	65536	1.18
`shift_17`	17	19	131072	1.73
`shift_18`	18	19	262144	3.19
`shift_19`	19	19	524288	7.84
`shift_20`	20	19	1048576	24.97
`shift`	21	19	2097152	71.33

In the second set of experiments, we test our algorithm to solve general case problems. We develop a program that takes an intersection pattern file and then write out the system of equations (10). The output system of equations can be fed into some specialized programs to solve for non-negative solution. We list the numbers of unknowns and the numbers of equations on some benchmarks in Table 2. We compare the number of unknowns obtained by our method to the number of unknowns of a naive method in which all 3^λ combinations of column patterns are taken as unknowns to set up equations. The number of unknowns generated by our method and the number of unknowns generated by the naive method are listed in the fourth column and the fifth column of Table 2, respectively. The ratio of the number of unknowns generated by our method to that generated by the naive method is listed in the sixth column. We can see that our algorithm greatly reduced the number of unknowns: for most of the benchmarks, our method can reduce more than 95% of unknowns. Thus, we believe that our proposed algorithm will greatly reduce the runtime to solve the general case problem compared to the naive method.

Table 2: Number of unknowns and number of equations for general case problems.

circuit	#cubes	#inputs	#unknowns			#equations
			our (a)	naive (b)	ratio (a/b)	
`luc`	6	8	66	729	0.091	32
`br2`	6	12	228	729	0.31	22
`tms`	8	8	262	6561	0.040	69
`prom2`	9	9	512	19683	0.026	265
`br1`	10	12	8108	59049	0.137	58
`vg2`	10	25	1294	59049	0.022	71
`exps`	12	8	4130	531441	0.008	399
`alu1`	12	12	4096	531441	0.008	1300
`exp`	14	8	69470	4782969	0.015	122
`newtpla`	14	15	127908	4782969	0.027	117

6. CONCLUSION AND FUTURE WORK

In this paper, we introduced a new problem, the λ-cube intersection problem: Given a set of numbers corresponding to an intersection pattern of a set of λ cubes, we are asked to synthesize a set of cubes to satisfy the given intersection pattern, or show that there is no solution to the problem. We provide a rigorous mathematical treatment to this problem and derive a necessary and sufficient condition for the existence of a set of cubes to satisfy the given intersection pattern. The problem reduces to check whether a set of linear equalities and inequalities has a non-negative integer solution.

As we mentioned in the introduction, a solution to the λ-cube intersection problem is an important step in solving the arithmetic two-level minimization problem. We are interested in the arithmetic two-level minimization problem because it applies to synthesis for probabilistic computation. We note that a solution to the problem could also be useful for generating weighted random testing patterns in built-in self-test (BIST) [6].

In future work, we will apply the techniques proposed in this paper to develop a general solution to the arithmetic two-level minimization problem. We will also study the special structure of the set of equations we derived in this paper; we will propose an efficient way to find a non-negative solution to these equations.

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